

ON THE NUMBER OF ZEROS OF DIAGONAL FORMS

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ABSTRACT. In this paper, we study the number of zeros of the equation $X_1^e + X_2^e + \cdots + X_n^e \equiv 0 \pmod{p}$, where $e > 1$ is a positive integer and $p \equiv 1 \pmod{e}$ is a prime. For $e = 5, 7$, we compute the recurrence satisfied by these numbers, using the generalized Jacobi sums.

1. INTRODUCTION

Let $e > 1$ be a positive integer and let p be a prime, $p \equiv 1 \pmod{e}$. We consider the number of solutions of the equation

$$X_1^e + X_2^e + \cdots + X_n^e = 0$$

in the finite field \mathbb{F}_p . Let $T_e(n)$ denote the number of such solutions. In the case of $e = 3$, Chowla, Cowles, and Cowles proved in [3], using Gauss sums, that for $p \equiv 1 \pmod{3}$

$$\sum_{n=1}^{\infty} T_3(n)x^n = \frac{x}{1-px} + \frac{x^2(p-1)(2+dx)}{1-3px^2-pdx^3},$$

where d is uniquely determined by

$$4p = d^2 + 27b^2 \quad \text{and} \quad d \equiv 1 \pmod{3}.$$

Myerson settled the case of $e = 4$ in the same manner in [11]. It is easy to do the cases $e = 3, 4$ and 6 , because the associated Gauss sum relates to the imaginary quadratic field. In [12], Richman gives a formula for the generating function in terms of the period polynomial of the Gauss sums, but it is difficult to find a formula for this polynomial.

In this paper, we shall consider an algorithm to obtain the recurrence for $T_e(n)$, using generalized Jacobi sums and certain quadratic partitions of p in some cases. But we cannot express the recurrence in general form, because the partition is very intricate except when $e = 3, 4, 6$ and 8 . For $e = 5$ and 7 , if $p < 1000$, we give the recurrence explicitly in Tables 2 and 5, and they may suggest some interesting conjectures. Our method can be applied in the case that the usual Jacobi sums or the cyclotomic numbers of order e are known explicitly,

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that is, $e = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 20, 24$ and 30 (see [1, 2, 4, 6–10, 13, 14]).

2. RECURRENCE

In this section, we show how to obtain the recurrence for $T_e(n)$, using generalized Jacobi sums. The fact that the $T_e(n)$ satisfy a recurrence was already proved by Chowla et al. in [3] and Richman in [12].

Now, since

$$\sum_{t=0}^{p-1} \zeta_p^{ta} = \begin{cases} p & \text{if } a \equiv 0 \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\zeta_p = \exp(\frac{2\pi i}{p})$, we have

$$pT_e(n) = \sum_{t=0}^{p-1} \sum_{x_1=0}^{p-1} \sum_{x_2=0}^{p-1} \dots \sum_{x_n=0}^{p-1} \zeta_p^{t(x_1^e + x_2^e + \dots + x_n^e)} = \sum_{t=0}^{p-1} \left(\sum_{x_1=0}^{p-1} \zeta_p^{tx^e} \right)^n = \sum_{t=0}^{p-1} G(t)^n,$$

where $G(t)$ is the Gauss sum of order e :

$$G(t) = \sum_{x_1=0}^{p-1} \zeta_p^{tx^e}.$$

We take Δ in \mathbb{F}_p such that Δ^d is not an e th power for any d ($0 < d < e$). Then, the multiplicative group \mathbb{F}_p^* of \mathbb{F}_p has the following coset decomposition:

$$\mathbb{F}_p^* = (\mathbb{F}_p^*)^e + \Delta(\mathbb{F}_p^*)^e + \Delta^2(\mathbb{F}_p^*)^e + \dots + \Delta^{e-1}(\mathbb{F}_p^*)^e.$$

Notice that each coset has $\frac{p-1}{e}$ elements and that $G(u) = G(v)$ for any $u, v \in \Delta^j(\mathbb{F}_p^*)^e$ ($j = 0, 1, \dots, e-1$). Since $G(0) = p$, we have

$$(1) \quad pT_e(n) = \frac{p-1}{e} \sum_{j=0}^{e-1} G(\Delta^j)^n + p^n.$$

Thus, $G(\Delta^j)$ ($j = 0, 1, \dots, e-1$) are roots of the polynomial

$$(2) \quad F_e(x) = x^e + y_1x^{e-1} + y_2x^{e-2} + \dots + y_e,$$

where $y_i = -\frac{1}{i} \sum_{j=0}^{i-1} y_j s_{i-j}$ with $y_0 = 1$ and $s_i = \sum_{j=0}^{e-1} G(\Delta^j)^i$. Since $s_i \equiv 0 \pmod{p}$ for any $i \geq 1$, we have

$$(3) \quad y_i \equiv 0 \pmod{p} \quad \text{for } 1 \leq i \leq e.$$

From (2), we see that the numbers $T_e(n)$ satisfy the recurrence

$$(pT_e(n+e) - p^{n+e}) + y_1(pT_e(n+e-1) - p^{n+e-1}) + \dots + y_e(pT_e(n) - p^n) = 0,$$

that is,

$$(4) \quad T_e(n+e) + y_1T_e(n+e-1) + \dots + y_eT_e(n) - p^{n-1}F_e(p) = 0.$$

Therefore, we have only to find $T_e(1), T_e(2), \dots, T_e(e)$. It is easy to check that

$$T_e(1) = 1, \quad T_e(2) = \begin{cases} 1 + e(p - 1) & \text{if } -1 \text{ is an } e\text{th power in } \mathbb{F}_p, \\ 1 & \text{otherwise} \end{cases}$$

and

$$y_1 = 0, \quad y_2 = \begin{cases} -pe(e - 1)/2 & \text{if } -1 \text{ is an } e\text{th power in } \mathbb{F}_p, \\ pe/2 & \text{otherwise.} \end{cases}$$

Next, we will represent $T_e(n)$, using the multiplicative character on \mathbb{F}_p^* . From now on, we denote a multiplicative character by χ . We call the character χ_0 defined by $\chi_0(a) = 1$ for all $a \in \mathbb{F}_p^*$ the trivial character. It is useful to extend the domain of definition of a multiplicative character to all of \mathbb{F}_p . If $\chi \neq \chi_0$ we do this by defining $\chi(0) = 0$. For χ_0 , we define $\chi_0(a) = 1$ for all $a \in \mathbb{F}_p$.

Since

$$(5) \quad \#\{x \in \mathbb{F}_p | x^e = a\} = \sum_{\chi^e=1} \chi(a) = \begin{cases} e & \text{if } a \text{ is an } e\text{th power in } \mathbb{F}_p, \\ 1 & \text{if } a = 0, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$T_e(n) = \sum_{\chi_1^e=1} \sum_{\chi_2^e=1} \cdots \sum_{\chi_n^e=1} \sum_{a_1+\cdots+a_n=0} \chi_1(a_1)\chi_2(a_2)\cdots\chi_n(a_n)$$

with $a_1, \dots, a_n \in \mathbb{F}_p$. If $a_n = 0$, then $\chi_n = \chi_0$, and the sum is $T_e(n - 1)$. If $a_n \neq 0$, then the sum equals

$$\begin{aligned} & \sum_{a_n=1}^{p-1} \sum_{\chi_1} \cdots \sum_{\chi_n} \chi_1 \cdots \chi_{n-1}(-1)\chi_1 \cdots \chi_n(a_n) \sum_{a_1+\cdots+a_{n-1}=1} \chi_1(a_1) \cdots \chi_{n-1}(a_{n-1}) \\ &= (p - 1) \sum_{\chi_1} \cdots \sum_{\chi_{n-1}} \chi_1 \cdots \chi_{n-1}(-1) \sum_{a_1+\cdots+a_{n-1}=1} \chi_1(a_1) \cdots \chi_{n-1}(a_{n-1}). \end{aligned}$$

Let $S_e(n)$ denote

$$\sum_{\chi_1} \cdots \sum_{\chi_n} \chi_1 \cdots \chi_n(-1) J_e(\chi_1, \dots, \chi_n),$$

where

$$(6) \quad J_e(\chi_1, \dots, \chi_n) = \sum_{a_1+\cdots+a_n=1} \chi_1(a_1) \cdots \chi_n(a_n)$$

is the generalized Jacobi sum. Hence, we have

$$(7) \quad T_e(n) = T_e(n - 1) + (p - 1)S_e(n - 1),$$

and we need to calculate sums of Jacobi sums $S_e(2), \dots, S_e(e - 1)$:

Remark. Since $G(\Delta) = \sum_a \#\{x \in \mathbb{F}_p | x^e = a\} \zeta_p^{\Delta a}$, from (5) we have $G(\Delta) \equiv 1 \pmod{e}$ in $\mathbb{Z}[\zeta_p]$, for $\Delta \not\equiv 0 \pmod{p}$. Then $e^e | F_e(p)$.

3. PROPERTIES OF GENERALIZED JACOBI SUMS

In this section, we will study the properties of generalized Jacobi sums. We can calculate them from usual Jacobi sums $J_e(\chi_1, \chi_2)$ by the following proposition.

Proposition. *The generalized Jacobi sum $J_e(\chi_1, \dots, \chi_n)$ satisfies the following properties (a)–(e):*

- (a) $J_e(\chi_1, \dots, \chi_i, \dots, \chi_j, \dots, \chi_n) = J_e(\chi_1, \dots, \chi_j, \dots, \chi_i, \dots, \chi_n)$.
- (b) *If all of the χ_i ($1 \leq i \leq n$) are trivial, then*

$$J_e(\chi_1, \dots, \chi_n) = p^{n-1}.$$

- (c) *If some but not all of the χ_i are trivial, then*

$$J_e(\chi_1, \dots, \chi_n) = 0.$$

- (d) *If χ_1, \dots, χ_n and $\chi_1\chi_2$ are nontrivial for $n \geq 3$, then*

$$J_e(\chi_1, \chi_2, \chi_3, \dots, \chi_n) = J_e(\chi_1, \chi_2) J_e(\chi_1\chi_2, \chi_3, \dots, \chi_n).$$

- (e) *If χ_1, \dots, χ_n are nontrivial and $\chi_1\chi_2$ is trivial, then*

$$J_e(\chi_1, \chi_2, \chi_3, \dots, \chi_n) = p J_e(\chi_3, \dots, \chi_n)$$

for $n \geq 4$, and $J_e(\chi_1, \chi_2, \chi_3) = p$.

Proof. See [5, pp. 98–101].

4. THE CASE $e = 5$

For p a prime $\equiv 1 \pmod{5}$, we consider the pair of diophantine equations

$$(8) \quad 16p = a_1^2 + 50a_2^2 + 50a_3^2 + 125a_4^2,$$

$$(9) \quad a_1a_4 = a_3^2 - 4a_2a_3 - a_2^2, \quad a_1 \equiv 1 \pmod{5}.$$

Dickson proved in [4] that this simultaneous system has exactly four solutions, and if one of these is (a_1, a_2, a_3, a_4) , the other three are

$$(a_1, -a_2, -a_3, a_4), \quad (a_1, a_3, -a_2, -a_4) \quad \text{and} \quad (a_1, -a_3, a_2, -a_4).$$

On the other hand, we can choose a prime factor π of p in the unique factorization domain $\mathbb{Z}[\zeta_5]$ and a character χ of order 5 associated with π , such that

$$J_5(\chi, \chi) = -\pi_1\pi_4,$$

where $\pi_i = \sigma_i(\pi)$ and σ_i is the automorphism of $\mathbb{Q}(\zeta_5)$ defined by $\sigma_i(\zeta_5) = \zeta_5^i$ ($1 \leq i \leq 4$), and $\zeta_5 = \exp(\frac{2\pi i}{5})$. If

$$J_5(\chi, \chi) = c_1\zeta_5 + c_2\zeta_5^2 + c_3\zeta_5^3 + c_4\zeta_5^4,$$

then the four solutions of (8) and (9) correspond to $\sigma_i(J_5(\chi, \chi)) = J_5(\chi^i, \chi^i)$ ($1 \leq i \leq 4$) by

$$\begin{aligned} 4c_1 &= -a_1 + 2a_2 + 4a_3 + 5a_4, & 4c_2 &= -a_1 + 4a_2 - 2a_3 - 5a_4, \\ 4c_3 &= -a_1 - 4a_2 + 2a_3 - 5a_4, & 4c_4 &= -a_1 - 2a_2 - 4a_3 + 5a_4. \end{aligned}$$

This result was proved by Dickson in [4].

From properties of Jacobi sums (see [5, 9]):

$$(10) \quad J_e(\chi^i, \chi^j) = J_e(\chi^j, \chi^i) \\ = (-1)^{(p-1)j/e} J_e(\chi^{-i-j}, \chi^j) \quad (i, j, i + j \not\equiv 0 \pmod{p}),$$

$$(11) \quad J_e(\chi^i, \chi^{-i}) = -\chi(-1) = -1 \quad (i \not\equiv 0 \pmod{p}),$$

we have

$$\begin{aligned} J_5(\chi, \chi^3) &= J_5(\chi, \chi), & J_5(\chi^4, \chi^4) &= J_5(\chi^2, \chi^4) = \sigma_4(J_5(\chi, \chi)), \\ J_5(\chi^3, \chi^3) &= J_5(\chi^3, \chi^4) = \sigma_3(J_5(\chi, \chi)), \\ J_5(\chi^2, \chi^2) &= J_5(\chi, \chi^2) = \sigma_2(J_5(\chi, \chi)), \end{aligned}$$

and

$$J_5(\chi, \chi^4) = J_5(\chi^2, \chi^3) = -1.$$

Hence, we obtain the following results by direct calculation from the proposition.

Theorem 1. *If (a_1, a_2, a_3, a_4) is any solution of (8) and (9), then*

$$S_5(2) = p - 4 + 3a_1,$$

$$S_5(3) = p^2 + 36p + a_1^2 - 3a_1 - 125a_4^2,$$

and

$$S_5(4) = \frac{1}{8} \{ a_1^3 - 8a_1^2 - 625a_1a_4^2 + 392a_1p - 2500a_2a_3a_4 \\ + 1000a_4^2 + 8p^3 - 288p \}.$$

From (1), (2), (3), (4) and (7), we can determine the recurrence for $T_5(n)$. It is given by

$$\begin{aligned} T_5(n + 5) - 10pT_5(n + 3) + pY_3T_5(n + 2) + pY_4T_5(n + 1) \\ + pY_5T_5(n) - 5^5Zp^n = 0, \quad n \geq 1, \end{aligned}$$

where

$$Y_3 = y_3/p = -5a_1, \quad Y_4 = y_4/p = -(a_1^2 - 125a_4^2 - 4p)/4,$$

$$Y_5 = y_5/p = -(a_1^3 - 625a_1a_4^2 - 8a_1p - 2500a_2a_3a_4)/8 \quad \text{and} \quad Z = F_5(p)/3125p.$$

For example, for $p < 1000$, the values of $a_1, a_2, a_3, a_4, Y_3, Y_4, Y_5, Z$ and the first few values are given in Table 1, Table 2 and Table 3.

TABLE 1

p	a_1	a_2	a_3	a_4
11	1	0	-1	1
31	11	1	-2	1
41	-9	0	-3	-1
61	1	1	4	-1
71	-19	2	3	1
101	-29	2	-3	-1
131	11	1	6	1
151	-4	2	-2	-4
181	11	2	7	-1
191	41	3	4	-1
211	1	1	2	-5
241	16	4	-4	4
251	-4	2	6	4
271	31	1	8	1
281	11	3	-4	5
311	-49	0	-7	-1
331	61	2	-5	1
401	-29	3	10	1
421	-19	1	-8	-5
431	36	6	-6	4
461	1	2	9	5
491	-9	3	12	1
521	31	6	7	-5
541	-59	1	10	-1
571	-44	2	-10	-4
601	-79	4	7	1
631	-89	4	-5	-1
641	16	4	12	-4
661	1	0	-3	9
691	41	2	-11	5
701	-79	4	9	1
751	71	4	11	-1
761	-99	3	-6	-1
811	101	2	-7	1
821	31	4	15	-1
881	61	8	-9	5
911	76	6	-10	4
941	-109	5	6	1
971	-44	6	14	4
991	-59	8	11	5

TABLE 2

p	Y_3	Y_4	Y_5	Z
11	-5	210	89	1
31	-55	160	409	185
41	45	260	-981	711
61	-5	460	-1111	3707
71	95	60	-101	7141
101	145	-390	-271	30463
131	-55	660	4009	86773
151	20	3235	-596	155648
181	-55	910	-1691	323951
191	-205	-990	-1331	401125
211	-5	4960	-961	604481
241	-80	3385	3344	1033472
251	20	3735	9004	1220224
271	-155	310	9599	1658645
281	-55	5160	5659	1923223
311	245	-1290	-4361	2904781
331	-305	-2840	-6541	3714113
401	145	1110	-1471	8075491
421	95	5560	-31751	9819947
431	-180	3035	9684	10775808
461	-5	6210	30539	14139931
491	45	2510	6219	18223497
521	-155	5310	7349	23112547
541	295	-1490	-13981	26932571
571	220	2935	-44476	33444608
601	395	-4640	16729	41099279
631	445	-6590	31259	49981709
641	-80	5385	-30256	53171200
661	-5	15960	6989	60166491
691	-205	5260	65419	71870249
701	395	-4140	11329	76230859
751	-355	-2390	379	100371029
761	495	-8290	43839	106001415
811	-505	-8540	-43361	136615577
821	-155	3060	5399	143582539
881	-305	3660	32009	190513423
911	-380	-165	34364	217885696
941	545	-9990	60169	248389591
971	220	4935	17924	281602048
991	295	4510	-10531	305614349

TABLE 3

p	$T_5(2)$	$T_5(3)$	$T_5(4)$	$T_5(5)$
11	51	151	4051	19251
31	151	1951	63151	1422751
41	201	601	126201	2127001
61	301	3901	351301	14095501
71	351	1051	553351	20697251
101	501	1501	1465501	89442501
131	651	21451	2860651	303345251
151	751	21001	3960751	515445001
181	901	38701	7101901	1091506501
191	951	59851	8569951	1405510751
211	1051	45151	10333051	1984536751
241	1201	69601	15661201	3418872001
251	1251	60001	17576251	3954325001
271	1351	98551	22762351	5504402251
281	1401	88201	24179401	6276529001
311	1551	51151	34256551	9120099251
331	1651	169951	41383651	12338922751
401	2001	126001	70542001	25624970001
421	2101	153301	79823101	31259728501
431	2151	232201	86432151	34836579001
461	2301	213901	104169301	45161730501
491	2451	227851	127010451	58008735751
521	2601	319801	150048601	74096321001
541	2701	197101	170669701	84807904501
571	2851	250801	197849851	105612051001
601	3001	219001	233733001	129031755001
631	3151	229951	270462151	156743480251
641	3201	441601	277027201	169170752001
661	3301	438901	297828301	190917160501
691	3451	562351	346107451	228920385751
701	3501	325501	366418501	239528747501
751	3751	723751	447528751	320096381251
761	3801	353401	468885801	332484933001
811	4051	903151	565222051	435949431751
821	4101	750301	578309101	455370333501
881	4401	937201	712232401	604762334001
911	4551	1037401	789338551	691887833001
941	4701	578101	876131701	779199299501
971	4851	814801	949343851	886859651001
991	4951	806851	1008913951	961599300751

5. THE CASE $e = 7$

Similarly, for p a prime $\equiv 1 \pmod{7}$, we consider the triple of diophantine equations

$$(12) \quad 72p = 2a_1^2 + 42(a_2^2 + a_3^2 + a_4^2) + 343(a_5^2 + 3a_6^2),$$

$$(13) \quad 12(a_2^2 - a_4^2 + 2a_2a_3 - 2a_2a_4 + 4a_3a_4) + 49(3a_5^2 + 2a_5a_6 - 9a_6^2) + 56a_1a_6 = 0,$$

$$(14) \quad 12(a_2^2 - a_4^2 + 4a_2a_3 + 2a_2a_4 + 2a_3a_4) + 49(a_5^2 + 10a_5a_6 - 3a_6^2) + 28a_1(a_5 + a_6) = 0,$$

$$(15) \quad a_1 \equiv 1 \pmod{7}.$$

This simultaneous system has six nontrivial solutions in addition to the two trivial solutions $(-6b_1, \pm 2b_2, \pm 2b_2, \mp 2b_2, 0, 0)$, where b_1 and b_2 are given by

$$(16) \quad p = b_1^2 + 7b_2^2, \quad b_1 \equiv -1 \pmod{7}.$$

If $(a_1, a_2, a_3, a_4, a_5, a_6)$ is one of the six nontrivial solutions of (12)–(15), the other five nontrivial solutions are

$$\begin{aligned} & (a_1, -a_3, a_4, a_2, -(a_5 + 3a_6)/2, (a_5 - a_6)/2), \\ & (a_1, -a_4, a_2, -a_3, -(a_5 - 3a_6)/2, -(a_5 + a_6)/2), \\ & (a_1, -a_2, -a_3, -a_4, a_5, a_6), \\ & (a_1, a_3, -a_4, -a_2, -(a_5 + 3a_6)/2, (a_5 - a_6)/2), \\ & (a_1, a_4, -a_2, a_3, -(a_5 - 3a_6)/2, -(a_5 + a_6)/2). \end{aligned}$$

For some character χ of order 7, these solutions correspond to Jacobi sums

$$J_7(\chi^i, \chi^i) = \sigma_i(J_7(\chi, \chi)) \quad (1 \leq i \leq 6),$$

with

$$(17) \quad J_7(\chi, \chi) = c_1\zeta_7 + c_2\zeta_7^2 + c_3\zeta_7^3 + c_4\zeta_7^4 + c_5\zeta_7^5 + c_6\zeta_7^6,$$

where

$$(18) \quad \begin{aligned} 12c_1 &= -2a_1 + 6a_2 + 7a_5 + 21a_6, & 12c_2 &= -2a_1 + 6a_3 + 7a_5 - 21a_6, \\ 12c_3 &= -2a_1 + 6a_4 - 14a_5, & 12c_4 &= -2a_1 - 6a_4 - 14a_5, \\ 12c_5 &= -2a_1 - 6a_3 + 7a_5 - 21a_6, & 12c_6 &= -2a_1 - 6a_2 + 7a_5 + 21a_6, \end{aligned}$$

and σ_i is the automorphism of $\mathbb{Q}(\zeta_7)$ defined by $\sigma_i(\zeta_7) = \zeta_7^i$ ($1 \leq i \leq 6$). The trivial solutions correspond to Jacobi sums $\sigma_i(J_7(\chi, \chi^2))$ ($i = 1, 6$) with

$$J_7(\chi, \chi^2) = b_1 + b_2\sqrt{-7}.$$

These results were proved by Leonard and Williams in [6]. Since

$$\begin{aligned} J_7(\chi, \chi^5) &= J_7(\chi, \chi), & J_7(\chi^6, \chi^6) &= J_7(\chi^2, \chi^6), & J_7(\chi^2, \chi^2) &= J_7(\chi^2, \chi^3), \\ J_7(\chi^5, \chi^5) &= J_7(\chi^4, \chi^5), & J_7(\chi^3, \chi^3) &= J_7(\chi, \chi^3), \\ J_7(\chi^4, \chi^4) &= J_7(\chi^4, \chi^6), & J_7(\chi, \chi^4) &= J_7(\chi^2, \chi^4) = J_7(\chi, \chi^2), \\ J_7(\chi^3, \chi^5) &= J_7(\chi^3, \chi^6) = J_7(\chi^5, \chi^6) = \sigma_6(J_7(\chi, \chi^2)), & \text{and} \\ J_7(\chi, \chi^6) &= J_7(\chi^2, \chi^5) = J_7(\chi^3, \chi^4) = -1, \end{aligned}$$

from (10) and (11) we obtain the following results from the proposition.

Theorem 2. *If $(a_1, a_2, a_3, a_4, a_5, a_6)$ is any nontrivial solution of (12)–(15) with $a_1 \equiv 1 \pmod{7}$ and (b_1, b_2) is the solution of (16) with $b_1 \equiv -1 \pmod{7}$, then*

$$S_7(2) = p - 6 + 3a_1 + 12b_1,$$

$$S_7(3) = p^2 + p - 6 - S_7(2) + 66p + 2a_1^2 + 8a_1b_1 + 14U_1 - 28U_2,$$

$$S_7(4) = \frac{p^4 - 1}{p - 1} - 7 - S_7(2) - S_7(3) + \frac{5}{144} \{8a_1^3 + 144a_1^2b_1 + 2736a_1p - 168a_1U_2 + 8640b_1p + 1008b_1U_1 - 2016b_1U_2 - 4116V_1(0, 1) + 2401V_2(5, -9)\}$$

$$S_7(5) = \frac{p^5 - 1}{p - 1} - 7 - S_7(2) - S_7(3) - S_7(4) + \frac{1}{12} \{8a_1^3b_1 + 972a_1^2p + 4896a_1b_1p - 168a_1b_1U_2 + 4320b_1^2p - 4116b_1V_1(0, 1) + 2401b_1V_2(5, -9) + 9576p^2 + 5544pU_1 - 9072pU_2\},$$

and

$$S_7(6) = \frac{p^6 - 1}{p - 1} - 7 - S_7(2) - S_7(3) - S_7(4) - S_7(5) - \frac{1}{864} \{34a_1^4b_1 - 5688a_1^3p - 334368a_1^2b_1p + 1512a_1^2b_1U_1 + 504a_1^2b_1U_2 - 362880a_1b_1^2p + 24696a_1b_1V_1(0, 1) - 4802a_1b_1V_2(13, 27) - 2334528a_1p^2 + 77112a_1pU_1 + 99792a_1pU_2 - 5145120b_1p^2 - 1829520b_1pU_1 + 2116800b_1pU_2 + 2646b_1U_1^2 - 3087b_1W_1(-2, 15, 27) - 6174b_1W_2(4, 12, 9) - 10584b_1W_3 + 3630312pV_1(0, 1) - 24696pV_1(1, 0) - 1210104pV_1(1, -1) + 14406pV_2(11, -45) - 2117682pV_2(5, -9) - 1411788pV_2(1, 36)\},$$

where

$$U_1 = a_2^2 + a_3^2 + a_4^2, \quad U_2 = a_2a_3 - a_2a_4 - a_3a_4,$$

$$V_1(i, j) = (i - j)a_2a_3a_5 + (2i + j)a_2a_4a_5 - (i + 2j)a_3a_4a_5 + 3(i + j)a_2a_3a_6 + 3ja_2a_4a_6 + 3ia_3a_4a_6,$$

$$V_2(i, j) = ia_3^3 + ja_3^2a_6 - 9ia_5a_6^2 - ja_6^3,$$

$$W_1(i, j, k) = (i - j + k)a_2^2a_5^2 + 2(3i - j - k)a_2^2a_5a_6 + (9i + 3j + k)a_2^2a_6^2 + (i + j + k)a_3^2a_5^2 - 2(3i + j - k)a_3^2a_5a_6 + (9i - 3j + k)a_3^2a_6^2 + 4ia_4^2a_5^2 + 4ja_4^2a_5a_6 + 4ka_4^2a_6^2,$$

$$W_2(i, j, k) = -(i - j + k)a_2a_3a_5^2 - 2(3i - j - k)a_2a_3a_5a_6 - (9i + 3j + k)a_2a_3a_6^2 + (i + j + k)a_3a_4a_5^2 - 2(3i + j - k)a_3a_4a_5a_6 + (9i - 3j + k)a_3a_4a_6^2 + 4ia_2a_4a_5^2 + 4ja_2a_4a_5a_6 + 4ka_2a_4a_6^2,$$

$$W_3 = a_2^3a_3 - a_3^3a_4 - a_4^3a_2.$$

Note, that U_1, \dots, W_3 are invariants under the change of the solution of (12)–(15).

It is hard to check the results of this theorem, but we calculated them as follows. Let $\mathcal{E}_i = \sigma_i(J_7(\chi, \chi))$ ($1 \leq i \leq 6$) and $\mathcal{H}_i = \sigma_i(J_7(\chi, \chi^2))$ ($i = 1, 6$). First, we expressed $S_7(2), \dots, S_7(6)$ in terms of 10 polynomials:

$$\begin{aligned} &\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6, \\ &\mathcal{E}_1^2 + \mathcal{E}_2^2 + \mathcal{E}_3^2 + \mathcal{E}_4^2 + \mathcal{E}_5^2 + \mathcal{E}_6^2, \\ &\mathcal{H}_1 + \mathcal{H}_2, \quad \mathcal{H}_1^2 + \mathcal{H}_2^2, \\ &\mathcal{E}_1\mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_4 + \mathcal{E}_4\mathcal{E}_1 + \mathcal{E}_3\mathcal{E}_5 + \mathcal{E}_5\mathcal{E}_6 + \mathcal{E}_6\mathcal{E}_3, \\ &\mathcal{E}_1\mathcal{E}_3 + \mathcal{E}_1\mathcal{E}_5 + \mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_6 + \mathcal{E}_4\mathcal{E}_5 + \mathcal{E}_4\mathcal{E}_6, \\ &\mathcal{E}_1^2\mathcal{E}_2 + \mathcal{E}_2^2\mathcal{E}_4 + \mathcal{E}_4^2\mathcal{E}_1 + \mathcal{E}_3\mathcal{E}_5^2 + \mathcal{E}_5\mathcal{E}_6^2 + \mathcal{E}_6\mathcal{E}_3^2, \\ &\mathcal{E}_1^2\mathcal{E}_3 + \mathcal{E}_1\mathcal{E}_5^2 + \mathcal{E}_2^2\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_6^2 + \mathcal{E}_4^2\mathcal{E}_5 + \mathcal{E}_4\mathcal{E}_6^2, \\ &\mathcal{E}_1^2\mathcal{H}_2 + \mathcal{E}_2^2\mathcal{H}_2 + \mathcal{E}_3^2\mathcal{H}_1 + \mathcal{E}_4^2\mathcal{H}_2 + \mathcal{E}_5^2\mathcal{H}_1 + \mathcal{E}_6^2\mathcal{H}_1, \\ &\mathcal{E}_1^3\mathcal{E}_2 + \mathcal{E}_2^3\mathcal{E}_4 + \mathcal{E}_4^3\mathcal{E}_1 + \mathcal{E}_3\mathcal{E}_5^3 + \mathcal{E}_5\mathcal{E}_6^3 + \mathcal{E}_6\mathcal{E}_3^3. \end{aligned}$$

Next, substituting (17) and (18) for them, they may be expressed as a polynomial in $p, a_1, b_1, U_1, \dots, W_3$.

Notice that b_2 is not needed in Theorem 2, but is determined by

$$J_7(\chi, \chi)J_7(\chi^2, \chi^2)J_7(\chi^4, \chi^4) = pJ_7(\chi, \chi^2).$$

The recurrence for $T_7(n)$ is given by

$$\begin{aligned} &T_7(n + 7) - 21pT_7(n + 5) + pY_3T_7(n + 4) + pY_4T_7(n + 3) + pY_5T_7(n + 2) \\ &+ pY_6T_7(n + 1) + pY_7T_7(n) - 7^7Zp^n = 0, \end{aligned}$$

where

$$\begin{aligned} Y_3 &= y_3/p = \frac{7}{3}(p - 6 - S_7(2)), \\ Y_4 &= y_4/p = \frac{7}{4}(p^2 + 127p - 6 - S_7(2) - S_7(3)), \\ Y_5 &= y_5/p = \frac{7}{5}(p^3 - 34p^2 + 35pS_7(2) + 211p - 6 - S_7(2) - S_7(3) - S_7(4)), \\ Y_6 &= y_6/p = \frac{7}{36}(6p^4 - 169p^3 - 28p^2S_7(2) - 8289p^2 + 14pS_7(2)^2 + 357pS_7(2) \\ &\quad + 189pS_7(3) + 1644p - 36) - \frac{7}{6}(S_7(2) + S_7(3) + S_7(4) + S_7(5)), \\ Y_7 &= y_7/p = \frac{1}{60}(60p^5 - 1459p^4 - 245p^3S_7(2) + 27941p^3 \\ &\quad - 31360p^2S_7(2) - 245p^2S_7(3) - 189864p^2 + 245pS_7(2)^2 \\ &\quad + 245pS_7(2)S_7(3) + 4704pS_7(2) + 3234pS_7(3) + 1764pS_7(4) \\ &\quad + 19464p - 360) - (S_7(2) + S_7(3) + S_7(4) + S_7(5) + S_7(6)), \\ Z &= F_7(p)/823543p. \end{aligned}$$

For example, for $p < 1000$, we have Table 4, Table 5 and Table 6.

TABLE 4

p	a_1	a_2	a_3	a_4	a_5	a_6	b_1
29	1	2	3	2	-1	1	-1
43	1	6	1	2	-1	1	6
71	15	2	0	-3	0	2	-8
113	-27	3	-6	-4	-3	1	-1
127	29	1	0	-12	1	1	-8
197	-13	6	-1	8	-5	1	13
211	-55	13	4	0	1	1	6
239	57	0	6	-11	0	2	-8
281	57	12	-6	7	0	2	13
337	-13	4	-15	-10	1	3	-15
379	-13	10	13	-12	-5	1	6
421	-55	4	-3	-18	-5	1	13
449	-41	19	0	10	5	1	-1
463	1	9	-22	0	5	1	20
491	-69	9	-20	-6	-3	1	-22
547	43	2	-15	0	-1	5	-22
617	-55	16	-6	1	-8	2	13
631	8	14	6	-18	4	4	-8
659	-27	9	-30	-4	-3	1	-22
673	22	12	30	-8	-4	4	-15
701	-125	20	3	-4	-1	1	-1
743	-27	3	20	-12	9	1	20
757	-27	13	4	14	-9	3	27
827	15	3	6	-26	9	1	-22
883	15	13	-32	-4	3	3	6
911	29	6	10	31	-2	4	-8
953	50	12	8	-28	4	4	-29
967	127	15	-6	20	-1	3	20

TABLE 5

p	Y_3	Y_4	Y_5	Y_6	Y_7	Z
29	21	2443	868	-45556	70643	289
43	-175	3031	32816	-54768	-919339	3563
71	119	8323	-83706	-263900	1928275	116297
113	217	5971	-20790	-28112	35447	2111327
127	21	10675	-36176	-1546524	8841251	4284689
197	-273	17731	288890	-1485568	-3682589	63088549
211	217	14203	-33824	-2902452	-15278033	97171097
239	-175	19495	11550	-4478096	22306759	206053139
281	-763	-3437	132090	947968	-2081087	547263017
337	511	15379	-588014	4909128	-7359899	1676462987
379	-77	50071	396886	-34381032	-386244419	3400759571
421	21	35371	356902	-7864584	-88796149	6427762907
449	315	20083	92890	-268408	-1427749	9501724321
463	-567	24787	676382	-6245820	-95682959	11390859329
491	1099	-11081	-265398	2764888	30094973	16362064007
547	315	57127	-1490398	-9087624	118738969	31322841539
617	21	51835	350042	-14093464	59902093	64730776477
631	168	75061	-1075760	-68095335	1009965704	74149915136
659	805	16555	-820470	-1070636	145798841	96475302113
673	266	80549	-1209824	-128631013	8403667999	109399460603
701	903	14203	-618296	-8719732	-12731797	140039937427
743	-371	85939	2575524	-72044588	-2482867129	198422927777
757	-567	68691	2678130	190232	-61712631	221974547877
827	511	85351	-2525964	-97022240	3008326775	378942918455
883	-273	36939	-292152	-8141140	93385725	561684402621
911	21	47719	426188	-16060912	-188652563	678164698651
953	462	119455	-3968524	-190379539	7358806846	890182377472
967	-1449	-19901	636986	19188708	135683491	969704283077

TABLE 6

p	$T_7(2)$	$T_7(3)$	$T_7(4)$	$T_7(5)$	$T_7(6)$	$T_7(7)$
29	197	589	87613	434141	48359081	401624189
43	295	4999	234319	7175071	279635161	10150704271
71	491	1471	651211	20725531	2039344721	123105907051
113	785	2353	2655409	123515281	21106338401	1967012774161
127	883	14995	3296035	258359851	35510978161	4191682116511
197	1373	61741	10524613	1623810021	305995801721	58947076303389
211	1471	24991	13272631	1842963991	433054373641	87522476780971
239	1667	74971	18167731	3410160391	797482745921	187210905127219
281	1961	170521	32651641	7108914121	1834761374321	499737158893561
337	2353	39985	49587217	12171117841	4427480228641	1458494959685905
379	2647	156115	61675615	20691045271	7845859876681	2963950323867163
421	2941	173461	88408741	31251603181	13312709379481	5566068700912861
449	3137	141121	110722753	39662795201	18423611731841	8181996374934593
463	3235	326635	119661235	47550132631	21475297314241	9869891508002527
491	3431	10291	151787791	54246811711	29020877711561	13952783047781971
547	3823	225499	183475111	88696102951	49112851733401	26778955559333251
617	4313	375145	264528265	144650373561	89691572093201	55166427521113833
631	4411	352801	274306411	158014498321	100202011231771	63116105003804161
659	4607	207271	334602871	183749622211	125035320648281	81817099907716363
673	4705	376321	330874657	203920893121	138299815325281	92895803047928161
701	4901	220501	400619101	235137559301	170235370769801	118536720987233501
743	5195	670027	443198827	306461088571	226744647572801	168262868716824967
757	5293	756757	476232373	331806309661	249105000979801	188234683135268869
827	5783	503035	611394463	464013218971	387427058908793	319850098013903395
883	6175	882883	767977687	611288203051	538121706358825	474064223649480703
911	6371	821731	835699411	688230705891	628767383411441	571610334899603091
953	6665	719713	914853913	821254899921	786651279340361	749072885603310465
967	6763	1534975	1032915787	894255059251	849085922746321	818225339860370275

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